

# Monad constructions of asymptotically stable bundles

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## Abstract

Motivated by gauge theory on  $G_2$ –manifolds, we produce several examples of bundles satisfying an ‘asymptotic’ stability condition over a divisor ‘at infinity’ over certain Fano 3-folds with exceptional holonomy studied by A. Kovalev. Such bundles are known to parametrise solutions of the Yang-Mills equation over the compact  $G_2$ –manifolds obtained from the initial Fanos by a twisted connected sum operation. One of our tools is a generalisation of Hoppe’s stability criterion to holomorphic bundles over smooth projective varieties  $X$  with  $\text{Pic } X \simeq \mathbb{Z}^l$ , a result which may be of independent interest.

## 1 Introduction

This paper presents a cohomological method to construct examples of holomorphic bundles, over certain Fano 3-folds, having the property of *asymptotic stability* (see below) over a distinguished anticanonical divisor, said to be ‘at infinity’ for geometrical reasons. It is based on the theory of instanton monads developed in [Jar<sub>1</sub>]. Indeed, our motivation comes from gauge theory in higher dimensions, particularly from the interplay between special holonomy in Calabi-Yau 3-folds and  $G_2$ –manifolds. In [Kov<sub>1</sub>, Kov<sub>2</sub>], A. Kovalev constructs new families of compact  $G_2$ –manifolds by proving a version of the Calabi conjecture for certain noncompact 3-folds, then performing a ‘twisted gluing’. Subsequently, the second named author established the existence of  $G_2$ –instantons over such manifolds, first by solving a suitable Hermitian Yang-Mills problem for 3-folds with tubular ends [SaE<sub>0</sub>, SaE<sub>1</sub>], then by gluing such solutions compatibly with Kovalev’s construction [SaE<sub>2</sub>]. The concept of asymptotic stability then emerges as the natural boundary condition for that analytical problem.

A *base manifold* for our purposes is a compact, simply-connected Kähler 3-fold  $(\bar{W}, \bar{\omega})$  carrying a  $K3$ -divisor  $D \in |-K_{\bar{W}}|$  with holomorphically trivial normal bundle  $\mathcal{N}_{D/\bar{W}}$  such that the complement  $W \doteq \bar{W} \setminus D$  has  $\pi_1(W)$  finite. Topologically  $W$  looks like a compact manifold  $W_0$  with boundary  $D \times S^1$  and a cylindrical end attached:

$$\begin{aligned} W &= W_0 \cup W_\infty \\ W_\infty &\simeq D \times S^1_\alpha \times (\mathbb{R}_+)_s. \end{aligned} \tag{1}$$

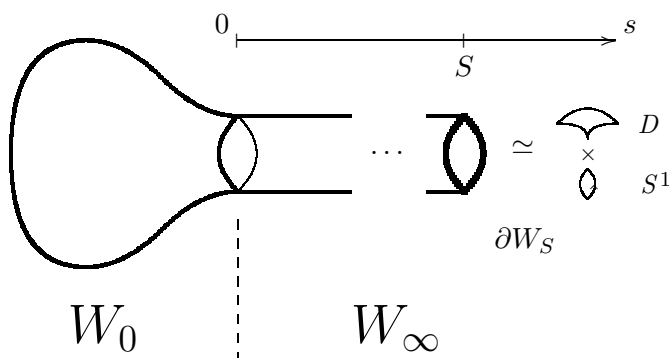


Figure 1: Asymptotically cylindrical Calabi-Yau 3-fold  $W$

For such base manifolds Kovalev has proved an ‘asymptotically cylindrical’ Calabi conjecture [Kov<sub>1</sub>, Theorem 2.2], as well as a gluing theorem for suitable pairs, truncated ‘far’ down the cylindrical ends. This yielded several new topological types of Riemannian 7-manifolds with holonomy group exactly  $G_2$ .

Concretely, Kovalev obtains asymptotically cylindrical Fano 3-folds from several types of initial Fano varieties, including:

- (i)  $V = \mathbb{P}^3$ .
- (ii)  $V \xrightarrow[D]{2:1} \mathbb{P}^3$ .
- (iii)  $V \subset \mathbb{P}^4$ ,  $\deg(V) = 2$ .
- (iv)  $V \subset \mathbb{P}^4$ ,  $\deg(V) = 3$ .
- (v)  $V = \mathbb{P}^2 \times \mathbb{P}^1$ .
- (vi)  $V = X_{22}$ , a certain prime subvariety of  $\mathbb{P}^{13}$ .

As to the gauge-theoretic data [SaE<sub>1</sub>], let  $z = e^{-s+i\alpha}$  be the holomorphic coordinate along the tube and denote  $D_z$  the corresponding  $K3$  component of the boundary. A bundle  $\mathcal{E} \rightarrow W$  is called *asymptotically stable* (or *stable at infinity*) if it is the restriction of an indecomposable holomorphic vector bundle  $\mathcal{E} \rightarrow \bar{W}$  such that  $\mathcal{E}|_D$  is stable (hence also  $\mathcal{E}|_{D_z}$  for  $|z| < \delta$ ). Such a bundle admits a smooth Hermitian ‘reference metric’  $H_0$ , with the property that  $H_0|_{D_z}$  are the corresponding HYM metrics on  $\mathcal{E}|_{D_z}$ ,  $0 \leq |z| < \delta$ , and which has ‘finite energy’, in a suitable sense.

Our crucial motivation is the fact that, given an asymptotically stable bundle with reference metric  $(\mathcal{E}, H_0)$ , there exists a nontrivial smooth solution to the  $G_2$ –instanton equation on  $p_1^*\mathcal{E} \rightarrow W \times S^1$  [*ibid.*, Theorem 59]. Moreover, such solutions can be glued, according to the ‘twisted connected sum’, to produce a  $G_2$ –instanton over the resulting *compact* 7-manifold with holonomy  $G_2$  [SaE<sub>2</sub>]. In other words, asymptotically stable bundles parametrise (a possibly proper subset of) solutions to the 7-dimensional Yang-Mills equation.

The present paper is organized as follows. We first establish a generalisation of the so-called *Hoppe criterion*, which gives a sufficient condition for a bundle over a projective variety to be stable in terms of the vanishing of certain cohomologies; this is one of our main tools in the paper. The other important tool, discussed in Section 3, is the construction of bundles via monads. Section 4 is dedicated to prove that the monads considered in Section 3 are indeed examples of asymptotically stable bundles over all of the initial Fano varieties in Kovalev’s list, except for case (vi), which was settled by Mukai in [Muk]. We complete the paper with a discussion of degenerations of asymptotically stable bundles to torsion-free sheaves.

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## 2 Generalised Hoppe Criterion

Let  $X$  be a nonsingular projective variety, and let  $E \rightarrow X$  be a holomorphic vector bundle of rank  $r$  and  $c_1(E) = 0$ . It is well-known that if  $E$  is (slope) stable then  $h^0(E) = 0$ , i.e. it has no holomorphic sections.

The so-called *Hoppe criterion* provides a partial converse to this result. More precisely, assume that  $\text{Pic}(X) \simeq \mathbb{Z}$  (such varieties are called *cyclic*), and recall that for any rank  $r$  holomorphic vector bundle  $E \rightarrow X$ , there is a uniquely determined integer  $k_E$  such that  $-r + 1 \leq c_1(E(k_E)) \leq 0$ . We set  $E_{\text{norm}} := E(-k_E)$ ; we say that  $E$  is *normalized* if  $E = E_{\text{norm}}$ . We then have the following criterion, c.f. [H, Lemma 2.6].

**Proposition 1.** *Let  $E$  be a rank  $r$  holomorphic vector bundle over a cyclic projective variety  $X$ . If  $H^0((\wedge^q E)_{\text{norm}}) = 0$  for  $1 \leq q \leq r - 1$ , then  $E$  is (slope) stable. If  $H^0((\wedge^q E)_{\text{norm}}(-1)) = 0$  for  $1 \leq q \leq r - 1$ , then  $E$  is (slope) semistable.*

The goal of this Section is to generalise the Hoppe criterion for a wider class of projective varieties. To be precise, a projective variety  $X$  will be said to be *polycyclic* if  $\text{Pic}(X) \simeq \mathbb{Z}^l$  for some  $l \geq 1$ . Writing  $\Upsilon_i \rightarrow X$ ,  $1 \leq i \leq l$ , for the corresponding generators and  $p \in \mathbb{Z}^l$  for a vector of multiplicities, one defines

$$\mathcal{O}_X(p) = \mathcal{O}_X(p_1, \dots, p_l) \doteq \Upsilon_1^{\otimes p_1} \otimes \dots \otimes \Upsilon_l^{\otimes p_l}.$$

Accordingly, given any other bundle  $E \rightarrow X$ , its (poly)twist is denoted

$$E(p) = E(p_1, \dots, p_l) \doteq E \otimes \mathcal{O}_X(p_1, \dots, p_l).$$

Set  $[h_i] := c_1(\Upsilon_i) \in H^2(X, \mathbb{Z})$ . Given a torsion-free coherent sheaf  $F$  of rank  $s$  over  $X$ , we have

$$\det F \doteq \det (\wedge^s F)^{\vee\vee} = \mathcal{O}_X(p_1, \dots, p_l),$$

where  $[c_1(F)] = p_1[h_1] + \dots + p_l[h_l]$ .

Now fix an ample line bundle  $L \rightarrow X$ , and set  $n := \dim X$ . As usual, we define the degree of  $F$  with respect to  $L$  as follows:

$$\deg_L F \doteq c_1(F) \cdot L^{n-1}.$$

Let  $\delta_L$  be the linear functional on the lattice  $\mathbb{Z}^l$  defined as follows:

$$\delta_L(p_1, \dots, p_l) := \deg_L \mathcal{O}_X(p_1, \dots, p_l).$$

Let us now consider a few examples of varieties which satisfy the requirements above.

**Example 2** (Cartesian product). *Consider the Cartesian product  $X = \mathbb{P}^n \times \mathbb{P}^m$  and let  $\pi_1$  and  $\pi_2$  are the two canonical projections. Set, as usual:*

$$\mathcal{O}_X(p_1, p_2) := \pi_1^* \mathcal{O}_{\mathbb{P}^n}(p_1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^m}(p_2),$$

*and set  $L := \mathcal{O}_X(1, 1)$ . Then given a torsion-free sheaf  $F$  with  $\det F = \mathcal{O}_X(p_1, p_2)$ , one has*

$$\deg_L F = \frac{n(n+1) \cdots (n+m+1)}{m!} \left( p_1 + \frac{m}{n} p_2 \right). \quad (2)$$

**Example 3** (Hirzebruch surfaces). *On  $X = \Sigma_a \doteq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1})$  one has  $\text{Pic}(X) = \mathbb{Z}.S_a \oplus \mathbb{Z}.H$ , with intersection form  $\begin{pmatrix} -a & 1 \\ 1 & 0 \end{pmatrix}$ . Recall  $\mathcal{O}_X(p, q)$  is ample if and only if  $p > 0$  and  $q > ap$ , so we may adopt  $L \doteq \mathcal{O}_X(1, a+1)$ . Thus if  $\det F = \mathcal{O}_X(p_1, p_2)$ , we have*

$$\deg_L F = (a+1)p_1 + p_2 - ap_1 = p_1 + p_2.$$

**Example 4** (Blow-up of  $\mathbb{P}^2$  at  $l$  distinct points). *On  $X := \tilde{\mathbb{P}}^2(l)$ , with exceptional divisors  $E_1, \dots, E_l$ , one has  $\text{Pic}(X) = \mathbb{Z}.E_1 \oplus \cdots \oplus \mathbb{Z}.E_l \oplus \mathbb{Z}.H$  such that  $E_k \cdot E_j = -\delta_{kj}$ ,  $E_k \cdot H = 0$  and  $H^2 = 1$ . Take  $L = \mathcal{O}_X(-1, \dots, -1, l+1)$  as the ample line bundle, thus if  $\det F = \mathcal{O}_X(p_1, \dots, p_{l+1})$ , we have:*

$$\deg_L F = p_1 + \cdots + p_l + (l+1)p_{l+1}.$$

**Remark 5.** *One easily adapt the examples above and take  $X$  to be the cartesian product of polycyclic varieties, e.g.  $X = \mathbb{P}^1 \times \Sigma_a$  or  $X = \mathbb{P}^1 \times \text{Bl}_z \mathbb{P}^2$ .*

Slope stability will be measured with respect to  $L$ . To be precise, the  $L$ -slope of a torsion-free sheaf  $F$  on  $X$  is given by

$$\mu_L(F) = \frac{\deg_L F}{\text{rank } F}$$

and a torsion-free sheaf  $E$  is said to be  $L$ -stable if every subsheaf  $F \hookrightarrow E$  satisfies  $\mu_L(F) < \mu_L(E)$ .

Let us now define the normalisation of a torsion-free sheaf  $E$  on  $X$  with respect to  $L$ . First, set  $d = \deg_L(\mathcal{O}_X(1, 0, \dots, 0))$ . Since

$$\deg_L(E(-k, 0, \dots, 0)) = \deg_L(E) - lk \cdot \text{rank}(E)$$

there is a unique integer  $k_E := \lceil \mu_L(E)/d \rceil$  such that

$$1 - d \cdot \text{rank}(E) \leq \deg_L(E(-k_E, 0, \dots, 0)) \leq 0;$$

the twisted bundle  $E_{L\text{-norm}} := E(-k_E, 0, \dots, 0)$ , is called the *L-normalisation* of  $E$ .

**Proposition 6** (Generalised Hoppe criterion). *Let  $E \rightarrow X$  be a rank  $r \geq 2$  holomorphic vector bundle over a polycyclic variety  $X$  of Picard rank  $l$ . If  $H^0((\Lambda^s E)_{L\text{-norm}}(\vec{p})) = 0$  for every  $1 \leq s \leq r-1$  and every  $\vec{p} \in \mathbb{Z}^l$  such that  $\delta_L(\vec{p}) \leq 0$ , then  $E$  is  $L$ -stable.*

*Proof.* Let  $F$  be a torsion free subsheaf of  $E$  with rank  $s$  and  $\det F = \mathcal{O}_X(\vec{q})$ . The inclusion  $\iota : F \hookrightarrow E$  induces a nontrivial section of  $H^0(\Lambda^s E(-\vec{q}))$ . Since  $\mu_L(\Lambda^s E) = s\mu_L(E)$ , we have that  $\Lambda^s E(-\vec{q}) = (\Lambda^s E)_{L\text{-norm}}(k_s - q_1, -q_2, \dots, -q_l)$  where  $k_s = \lceil s\mu_L(E)/d \rceil$ .

Hence, by hypothesis,  $0 < \delta_L(k_s - q_1, -q_2, \dots, -q_l) = dk_s - \deg_L(F)$ , thus  $s\mu_L(F) < dk_s$ , thus

$$\frac{s\mu_L(F)}{d} < \left\lceil \frac{s\mu_L(E)}{d} \right\rceil.$$

It follows that  $\mu_L(F) < \mu_L(E)$ . Hence  $E$  is  $L$ -stable.  $\square$

### 3 Monads over products of projective spaces

Recall that a *monad* on  $X$  is a complex of locally free sheaves

$$M_\bullet : M_0 \xrightarrow{\alpha} M_1 \xrightarrow{\beta} M_2$$

such that  $\beta$  is locally right-invertible,  $\alpha$  is locally left-invertible. The (locally free) sheaf  $E := \ker \beta / \text{img } \alpha$  is called the *cohomology* of  $M_\bullet$ .

Monads are a valuable tool in the theory of sheaves over projective varieties, and have been studied by many authors in the past 40 years. The paper [Jar1] is of particular interest to us; there, the first named author used the so-called *linear monads* to produce examples of stable bundles over cyclic varieties of dimension 3. More precisely, the following result is proved.

**Theorem 7.** *Let  $X$  be a cyclic nonsingular complex projective 3-fold with fundamental class  $h := c_1(\mathcal{O}_X(1))$ , and  $c \geq 1$  an integer; then a linear monad of the form*

$$0 \rightarrow \mathcal{O}_X(-1)^{\oplus c} \xrightarrow{\alpha} \mathcal{O}_X^{\oplus 2+2c} \xrightarrow{\beta} \mathcal{O}_X(1)^{\oplus c} \rightarrow 0 \quad (3)$$

*has the following properties:*

- the kernel bundle  $K := \ker \beta$  is a stable bundle with

$$\text{rank}(K) = c + 2, \quad c_1(K) = -c \cdot h, \quad c_2(K) = \frac{1}{2} (c^2 + c) \cdot h^2.$$

- the cohomology bundle  $E := \ker \beta / \text{img } \alpha$  is a stable bundle with

$$\text{rank}(E) = 2, \quad c_1(E) = 0, \quad c_2(E) = c \cdot h^2.$$

We will now see another instance of such construction of bundles using monads, obtaining stable vector bundles of rank  $r = 2$  over the product  $X = \mathbb{P}^n \times \mathbb{P}^m$ . The remainder of this *Section* is devoted to the proof of the following result.

**Theorem 8.** *Let  $X = \mathbb{P}^n \times \mathbb{P}^m$  with  $m \leq n$ , and set  $L := \mathcal{O}_X(1, 1)$ . Then monads of the form*

$$0 \rightarrow \mathcal{O}_X(-1, 0)^{\oplus a} \xrightarrow{\alpha} \mathcal{O}_X^{\oplus b} \oplus \mathcal{O}_X(-1, 1)^{\oplus c} \xrightarrow{\beta} \mathcal{O}_X(0, 1)^{\oplus a} \rightarrow 0 \quad (4)$$

with  $a, b, c \in \mathbb{N}$  such that  $r := b + c - 2a = 2$ , have the following properties:

- the kernel bundle  $K := \ker \beta$  is a  $L$ -stable bundle of rank  $a + 2$  with

$$\deg_L(K) = \frac{n(n+1) \cdots (n+m+1)}{m!} \left( \left( \frac{m}{n} - 1 \right) c - \frac{m}{n} a \right);$$

- for  $c \geq a$ , the cohomology bundle  $E := \ker \beta / \text{img } \alpha$  is a  $L$ -stable bundle of rank 2 with

$$\deg_L(E) = \frac{n(n+1) \cdots (n+m+1)}{m!} \left( 1 - \frac{m}{n} \right) (a - c).$$

We begin with the following two lemmata.

**Lemma 9.** *If  $q_1 + q_2 < 0$ , then  $h^p(\mathcal{O}_X(q_1, q_2)^{\oplus k}) = 0$  for  $0 \leq p < n + m$  and all  $k \in \mathbb{N}$ .*

*Proof.* From the Künneth formula, we know that

$$H^p(\mathcal{O}_X(q_1, q_2)) = \bigoplus_{p_1 + p_2 = p} H^{p_1}(\mathcal{O}_{\mathbb{P}^n}(p_1)) \otimes H^{p_2}(\mathcal{O}_{\mathbb{P}^m}(p_2)). \quad (5)$$

If  $q_1 + q_2 < 0$ , then either  $H^0(\mathcal{O}_{\mathbb{P}^n}(p_1)) = 0$  or  $H^0(\mathcal{O}_{\mathbb{P}^m}(p_2)) = 0$ . Thus  $H^1(\mathcal{O}_X(q_1, q_2)) = 0$  and, in formula (5), it is enough to consider summands with  $1 \leq p_1, p_2 \leq p - 1$  when  $2 \leq p \leq n + m - 1$ . We then conclude that if  $q_1 + q_2 < 0$ , then  $H^p(\mathcal{O}_X(q_1, q_2)) = 0$  for  $0 \leq p \leq n_1 + n_2 - 1$ . Since  $\mathcal{O}_X(q_1, q_2)^{\otimes k} = \mathcal{O}_X(kq_1, kq_2)$ , we obtain the desired conclusion.  $\square$

**Lemma 10.** *Let  $A, B \rightarrow X$  be vector bundles canonically pulled back from bundles  $A' \rightarrow \mathbb{P}^n$  and  $B' \rightarrow \mathbb{P}^m$ ; then*

$$H^q(\Lambda^s(A \otimes B)) = \sum_{k_1 + \dots + k_s = q} \left\{ \bigoplus_{i=1}^s \left( \sum_{j=0}^s \sum_{m=0}^{k_i} H^m(\Lambda^j(A)) \otimes H^{k_i-m}(\Lambda^{s-j}(B)) \right) \right\}$$

*Proof.* Using the following facts:

$$\begin{aligned} H^q(A_1 \oplus \dots \oplus A_s) &= \sum_{k_1 + \dots + k_s = q} \left\{ \bigoplus_{i=1}^s H^{k_i}(A_i) \right\}, \\ H^q(A \otimes B) &= \sum_{m=0}^q H^m(A) \otimes H^{q-m}(B), \\ \Lambda^s(A \otimes B) &= \sum_{j=0}^s \Lambda^j(A) \otimes \Lambda^{s-j}(B). \end{aligned}$$

□

**Proof of Theorem 8:** The claims about rank and degree  $K$  and  $E$  are immediate applications of the splitting principle, using formula (2). Consider also the associated *canonical diagram*:

$$\begin{array}{ccccccc} & & 0 & & & & \\ & & \downarrow & & & & \\ & & \mathcal{O}_X(-1, 0)^{\oplus a} & & & & \\ & & \downarrow & & & & \\ 0 & \rightarrow & K & \longrightarrow & \mathcal{O}_X^{\oplus b} \oplus \mathcal{O}_X(-1, 1)^{\oplus c} & \longrightarrow & \mathcal{O}_X(0, -1)^{\oplus a} \rightarrow 0 \\ & & \downarrow & & & & \\ & & E & & & & \\ & & \downarrow & & & & \\ & & 0 & & & & \end{array}$$

Let us start with the stability of the kernel  $K$ ; since  $\deg_L(K) < 0$  (because  $m \leq n$ ), it is enough, by Proposition 6, to prove the following fact:

$$h^0(\Lambda^s K(-p_1, -p_2)) = 0, \quad \forall \left\{ \begin{array}{l} 1 \leq s \leq a+1 \\ p_1 + p_2 \leq 0 \end{array} \right\}.$$

Now, to any short exact sequence of locally free sheaves  $0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$  there is an associated long exact sequence given by successive symmetrisation of alternating powers:

$$0 \rightarrow \Lambda^s \mathcal{F} \rightarrow \Lambda^s \mathcal{G} \rightarrow \Lambda^{s-1} \mathcal{G} \otimes \mathcal{H} \rightarrow \dots \rightarrow \mathcal{G} \otimes S^{s-1} \mathcal{H} \rightarrow S^s \mathcal{H} \rightarrow 0, \quad \forall s \geq 1.$$



Twisting the horizontal string of the diagram by  $\mathcal{O}_X(-p_1, -p_2)$ , this implies

$$0 \rightarrow H^0(\Lambda^s K(-p_1, -p_2)) \rightarrow H^0(\Lambda^s \mathcal{G}) \rightarrow \dots$$

with  $\mathcal{G} = \mathcal{O}_X^{\oplus b}(-p_1, -p_2) \oplus \mathcal{O}_X(-1-p_1, 1-p_2)^{\oplus c}$ . By Lemma 10,  $H^0(\Lambda^s \mathcal{G})$  expands into a sum of terms  $\pi_1^* H^0(\Lambda^j \mathcal{O}_{X_1}(\tilde{p}_1)) \otimes \pi_2^* H^0(\Lambda^{1-j} \mathcal{O}_{X_2}(\tilde{p}_2))$ , where at least one of  $\tilde{p}_1, \tilde{p}_2$  in each case is strictly negative. So we have

$$h^0(\Lambda^s K(-p_1, -p_2)) = h^0(\Lambda^s \mathcal{G}) = 0 \quad (6)$$

and this proves the stability claim for  $K$ .

Let us now examine the cohomology bundle  $E$ . Since  $E$  has rank 2 and  $\deg_L(E) \leq 0$  (because  $c \geq a$ ), it is enough to check that  $E(-p_1, -p_2)$  has no sections whenever  $p_1 + p_2 \geq 0$ , by Proposition 6. This follows easily from (6), Lemma 9 and the short exact sequence

$$0 \rightarrow \mathcal{O}_X(-1-p_1, -p_2) \rightarrow K(-p_1, -p_2) \rightarrow E(-p_1, -p_2) \rightarrow 0.$$

This completes the proof.  $\square$

### 3.1 Existence of monads over $\mathbb{P}^2 \times \mathbb{P}^1$

The existence of monads of the form (4) over a given  $\mathbb{P}^n \times \mathbb{P}^m$  is a non-trivial matter, c.f. [Flø] where necessary and sufficient criteria for the existence of monads over  $\mathbb{P}^n$  are established. Motivated by Kovalev's list, we will now prove the existence of monads of the form (4) over  $\mathbb{P}^2 \times \mathbb{P}^1$ .

More generally, in the condition of Theorem 8 the maps  $\alpha$  and  $\beta$  in the monad (4) are of the following form

$$\alpha = \begin{pmatrix} A \\ B \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} C & D \end{pmatrix},$$

where

$$\begin{aligned} A &\in \text{Mat}_{\mathbb{C}}(b \times a) \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1)) & B &\in \text{Mat}_{\mathbb{C}}(c \times a) \otimes H^0(\mathcal{O}_{\mathbb{P}^m}(1)) \\ C &\in \text{Mat}_{\mathbb{C}}(a \times b) \otimes H^0(\mathcal{O}_{\mathbb{P}^m}(1)) & D &\in \text{Mat}_{\mathbb{C}}(a \times c) \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \end{aligned}$$

satisfy the following ADHM-type equation

$$CA + DB = 0.$$

Now let  $X = \mathbb{P}^2 \times \mathbb{P}^1$ , with homogeneous coordinates  $[x_0 : x_1 : x_2]$  and  $[y_0 : y_1]$  on  $\mathbb{P}^2$  and  $\mathbb{P}^1$ , respectively, and consider the explicit monad

$$0 \rightarrow \mathcal{O}_X \xrightarrow{\alpha} \mathcal{O}_X(1, 0)^{\oplus 2} \oplus \mathcal{O}_X(0, 1)^{\oplus 2} \xrightarrow{\beta} \mathcal{O}_X(1, 1) \rightarrow 0 \quad (7)$$

given by

$$\alpha = \begin{pmatrix} x_0 \\ x_1 \\ y_0 \\ y_1 \end{pmatrix} \quad \text{and} \quad \beta = (y_0 \quad y_1 \quad -x_0 \quad -x_1) .$$

Twisting by  $\mathcal{O}_X(-1, 0)$  one obtains precisely an instanton monad of the form (4), with  $a = 1, b = c = 2$ , thus indeed  $b + c - 2a = 2$ .

## 4 Asymptotic stability over Kovalev's examples

Let  $E \rightarrow X^3$  be a holomorphic bundle over a Fano threefold, and  $D \subset X$  a divisor with  $\deg D \doteq d > 0$ , i.e.,  $D = \sigma_D^{-1}(0)$  for some section  $\sigma_D \in H^0(\mathcal{O}_X(d))$ . Recall from the Introduction that  $E$  is said to be *asymptotically stable* with respect to  $D$  if its restriction to  $D$  is stable.

We already know that to each case of Kovalev's list it is possible to assign a monad construction with stable cohomology bundle  $E$ , say. Furthermore, as we will now see, such bundles are indeed examples of asymptotically stable bundles with respect to a suitable divisor.

In order to accomplish this goal, we will require a few general facts. First, note that restriction sequence is given by:

$$0 \rightarrow E(-d) \xrightarrow{\sigma_D} E \xrightarrow{r_{X,D}} E|_D \rightarrow 0. \quad (8)$$

Moreover, we will also need the following two facts.

**Criterion 11.** *If  $F$  is a rank 2 bundle of degree 0 over a cyclic variety  $X$ , then  $F$  is stable if and only if  $h^0(F) = 0$ .*

Compare the above with [OSS, Lemma 1.2.5, p. 165] for the case  $X = \mathbb{P}^n$ ; the proof for  $X$  cyclic is the same.

**Definition 12.** *A line bundle  $L \rightarrow X$  over a projective variety of dimension  $n$  is said to be without intermediate cohomology (WIC) if*

$$H^i(L^{\otimes k}) = 0, \quad \forall \begin{cases} i = 1, \dots, n-1 \\ k \in \mathbb{Z} \end{cases} .$$

*A projective variety  $X$  is said to be WIC if the line bundle  $\mathcal{O}_X(1)$  is WIC.*

**Remark 13.** *It is not difficult to see, using Kodaira's vanishing theorem, that if  $X$  is cyclic and Fano, then the positive generator of  $\text{Pic } X$ , denoted  $\mathcal{O}_X(1)$ , is WIC. Furthermore, let  $X_1$  and  $X_2$  be cyclic WIC projective varieties, and set  $X = X_1 \times X_2$ ; then the line bundle  $\mathcal{O}_X(1, 1)$  is also WIC.*

Finally, recall also that complete intersection subvarieties of dimension at least 3 in  $\mathbb{P}^n$ ,  $n \geq 4$ , are cyclic and WIC.

#### 4.1 The $r = 2$ , cyclic case

Now let  $X$  be a nonsingular, cyclic Fano 3-fold, and let  $D \subset X$  be a cyclic divisor of degree  $d$ . It is easy to see that such framework includes items (i), (iii) and (iv) of Kovalev's list. The goal of this Section is to prove the following result.

**Proposition 14.** *Let  $(X, D)$  be as above. If  $E \rightarrow X$  arises from an instanton monad of the form (3), then  $E$  is asymptotically stable with respect to  $D$ .*

Note that since  $E$  can be obtained as the cohomology of a monad of the form (3), Theorem 7 guarantees that it is stable. Denoting  $K \doteq \ker \beta$  and twisting the monad by  $\mathcal{O}_X(-d)$ , the relevant data fit in the following canonical diagram:

$$\begin{array}{ccccccc}
& & 0 & & & & \\
& & \downarrow & & & & \\
& & \mathcal{O}_X(-(d+1))^{\oplus c} & & & & \\
& & \downarrow & & & & \\
0 & \rightarrow & K(-d) & \rightarrow & \mathcal{O}_X(-d)^{\oplus r+2c} & \rightarrow & \mathcal{O}_X(-(d-1))^{\oplus c} \rightarrow 0 \\
& & \downarrow & & & & \\
0 & \rightarrow & E(-d) & \rightarrow & E & \rightarrow & E|_D \rightarrow 0 \\
& & \downarrow & & & & \\
& & 0 & & & & 
\end{array}$$

*Proof.* In view of Criterion 11 and since  $h^0(E) = 0$ , it follows from the second horizontal string of the canonical diagram above that it is enough to show that:

$$h^1(E(-d)) = 0.$$

Note that  $h^0(\mathcal{O}_X(-k)) = 0$  and, since  $X$  is WIC (c.f. Remark 13),  $h^2(\mathcal{O}_X(k)) = 0$  for all  $k \in \mathbb{Z}$ . Therefore, it follows from the first horizontal string in the canonical diagram that  $h^1(K(-d)) = 0$ . Finally, one uses the vertical string of the canonical diagram to show that  $h^1(E(-d)) = 0$ , as desired.  $\square$

We have therefore provided many examples of rank 2 asymptotically stable bundles over varieties of type (i), (iii) and (iv) of Kovalev's list, taking  $D$  to be a generic anticanonical divisor  $D \in |-K_X|$ . The same examples can be pulled back to double covers of type (ii).

## 4.2 The polycyclic case

It remains for us to explore the possibility of generalising this method to cover case (v). We will work in the following more general framework. The first step is the following partial generalisation of Criterion 11.

**Lemma 15.** *Let  $X$  be a polycyclic variety  $X$ , and let  $L := \mathcal{O}_X(q_1, \dots, q_l)$  be an ample line bundle on  $X$ . If  $F \rightarrow X$  is a  $L$ -stable bundle with  $\deg_L(F) \leq 0$ , then  $h^0(F) = 0$ .*

*Proof.* It is easy to see that if  $h^0(F) \neq 0$ , then  $F$  is destabilised by the structure sheaf  $\mathcal{O}_X$ .  $\square$

Now let  $X = X_1 \times X_2$  be a product of cyclic WIC varieties, and let  $D \subset X$  be a divisor. We will say that  $D$  has *positive polydegree* if  $D \in |\mathcal{O}_X(d_1, d_2)|$  with  $d_1, d_2 \geq 0$ . Set  $L := \mathcal{O}_X(1, 1)$  and assume that  $\delta_L(-1, 1) > 0$ .

**Proposition 16.** *Let  $E \rightarrow X$  be a  $L$ -stable vector bundle obtained as the cohomology of a monad of the form*

$$0 \rightarrow \mathcal{O}_X(-1, 0)^{\oplus a} \xrightarrow{\alpha} \mathcal{O}_X^{\oplus b} \oplus \mathcal{O}_X(-1, 1)^{\oplus c} \xrightarrow{\beta} \mathcal{O}_X(0, 1)^{\oplus a} \rightarrow 0 \quad (9)$$

*such that  $c - a \leq 0$  and  $b + c - 2a = 2$ . If  $D \subset X$  is a polycyclic divisor of positive polydegree, then  $E|_D$  is  $\mathcal{O}_D(1, 1)$ -stable.*

*Proof.* The hypothesis imposed imply that the cohomology bundle  $E$  is a rank 2 bundle with nonpositive degree with respect to  $L$ , hence  $h^0(E) = 0$  by Criterion 15.

Denoting, as usual,  $(-\vec{d})$  for the twist by the sheaf  $\mathcal{O}_X(-d_1, -d_2)$ , the corresponding canonical diagram is:

$$\begin{array}{ccccccc} & & 0 & & & & \\ & & \downarrow & & & & \\ & & \mathcal{O}_X^{\oplus c}(-d_1 - 1, -d_2) & & & & \\ & & \downarrow & & & & \\ 0 & \rightarrow & K(-\vec{d}) & \rightarrow & \mathcal{O}_X(-\vec{d})^{\oplus b} \oplus \mathcal{O}_X^{\oplus c}(-d_1 - 1, -d_2 + 1) & \rightarrow & \mathcal{O}_X^{\oplus c}(-d_1, -d_2 - 1) \rightarrow 0 \\ & & \downarrow & & & & \\ 0 & \rightarrow & E(-\vec{d}) & \rightarrow & E & \rightarrow & E|_D \rightarrow 0 \\ & & \downarrow & & & & \\ & & 0 & & & & \end{array}$$

Since  $h^0(E) = 0$ , the bottom horizontal line tells us that it is enough to show that  $h^1(E(-\vec{d})) = 0$ . An argument similar to Lemma 9 guarantees that  $h^p(\mathcal{O}_X(q_1, q_2)) = 0$  for  $q_1 + q_2 < 0$  and  $0 \leq p \leq 2$ , thus

$$h^1(E(-\vec{d})) = h^1(K - \vec{d}) = h^2(\mathcal{O}_X^{\oplus c}(-1, 0)(-p - d)) = 0.$$

Hence, by Proposition 6,  $E|_D$  is stable with respect to  $\mathcal{O}_D(1, 1)$ .  $\square$

Together with the monad construction in the previous Section, we have provided many examples of rank 2 bundles over  $\mathbb{P}^2 \times \mathbb{P}^1$  which are asymptotically stable with respect to  $D \in |\mathcal{O}_D(2, 3)|$ , thus completing Kovalev's list.

## 5 Degeneration of asymptotically stable bundles

Another illustration of the usefulness of monads in gauge theory is the modelling of degenerating instanton sequences. Concretely, let us examine the task of producing a one-parameter family  $\{\mathcal{E}_\lambda\}_{\lambda>0}$  of asymptotically stable locally-free sheaves over some complex 3-fold  $W$  admissible by Kovalev's construction, say  $W = \mathbb{P}^3$ , such that  $\mathcal{E}_0 \doteq \lim_{\lambda \rightarrow 0} \mathcal{E}_\lambda$  is torsion-free.

### 5.1 Criteria on sheaf topology and degeneration locus

We consider a monad of locally-free sheaves of the form

$$0 \rightarrow V \otimes \mathcal{O}_X(-1) \xrightarrow{\alpha} W \otimes \mathcal{O}_X \xrightarrow{\beta} U \otimes \mathcal{O}_X(1) \rightarrow 0$$

with  $\alpha \in \text{Hom}(V, W) \otimes H^0(\mathcal{O}_X(1))$  injective and  $\beta \in \text{Hom}(W, U) \otimes H^0(\mathcal{O}_X(1))$  surjective, as sheaf maps. Its *degeneration locus* is the set

$$\Sigma = \{z \in \mathbb{P}^n \mid \ker \alpha_z \neq \{0\}\}.$$

Moreover, a *linear sheaf* is a coherent sheaf which is the cohomology of a linear monad. Then we have [Jar<sub>2</sub>, Prop. 4]:

**Criterion 17.** *Let  $\mathcal{E}$  be a linear sheaf; then*

1.  $\mathcal{E}$  is locally-free  $\Leftrightarrow \Sigma = \emptyset$ .
2.  $\mathcal{E}$  is reflexive  $\Leftrightarrow \Sigma$  is a subvariety with  $\text{codim } \Sigma \geq 3$ .
3.  $\mathcal{E}$  is torsion-free  $\Leftrightarrow \Sigma$  is a subvariety with  $\text{codim } \Sigma \geq 2$ .

All meaningful deductive claims in the sequel stem from this result.

## 5.2 Example over $W = \mathbb{P}^3$

For  $\lambda > 0$ , consider the family of linear monads over  $\mathbb{P}^3$ :

$$0 \rightarrow \mathcal{O}_X(-1) \xrightarrow{\alpha_\lambda} \mathcal{O}_X^{\oplus 4} \xrightarrow{\beta} \mathcal{O}_X(1) \rightarrow 0$$

with

$$\alpha_\lambda = \begin{pmatrix} z_1 \\ z_2 \\ \lambda z_3 \\ \lambda z_4 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} -z_2 & z_1 & -z_4 & z_3 \end{pmatrix}.$$

Clearly  $\Sigma_\lambda = \emptyset$ , hence the associated linear sheaf  $\mathcal{E}_\lambda$  is locally-free. Note that  $\text{rank}(\mathcal{E}_\lambda) = 2$ ,  $c_1(\mathcal{E}_\lambda) = 0$  and  $c_2(\mathcal{E}_\lambda) = 1$ , but more general topological types can also be arranged. It follows from Theorem 7 that  $\mathcal{E}_\lambda$  is stable, while Proposition 14 show that  $\mathcal{E}_\lambda$  is asymptotically stable for each  $\lambda > 0$ .

Now the limit  $\mathcal{E}_0$  is obviously still a linear sheaf. The new phenomenon of course is the fact that

$$\Sigma_0 = \{z_1 = z_2 = 0\} \subset \mathbb{P}^3$$

is a *curve*, hence  $\mathcal{E}_0$  is a properly torsion free sheaf. One can show that  $\mathcal{E}_0$  is properly semistable, see [Jar<sub>2</sub>, Proposition 14 and Example 4]. Its restriction to a canonical divisor  $D$  will also be a properly torsion free sheaf, since it intersects the degeneration locus at a single point.

Moreover,  $\mathcal{E}_0|_D$  is properly semistable; indeed, note that since we assume that  $\text{Pic}(D) = \mathbb{Z}$ , it is enough to check that  $h^0(\mathcal{E}_0(-1)|_D) = h^0(\mathcal{E}_0^*(-1)|_D) = 0$ , c.f. [Jar<sub>2</sub>, Lemma 13]. We already know that  $h^0(\mathcal{E}_0(-1)|_D) = 0$ . Since  $\mathcal{E}_0^*$  is a subsheaf of  $K^*$ , we conclude that  $h^0(K^*(-1)|_D) = 0$  implies that  $h^0(\mathcal{E}_0^*(-1)|_D) = 0$ ; to check that  $h^0(K^*(-1)|_D) = 0$ , simply consider the restriction sequence

$$0 \rightarrow K^*(-5) \rightarrow K^*(-1) \rightarrow K^*(-1)|_D \rightarrow 0$$

and note that  $h^0(K^*(-1)) = h^1(K^*(-5)) = 0$ , which follows from the sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow \mathcal{O}_{\mathbb{P}^3}^{\oplus n+1} \rightarrow K^* \rightarrow 0.$$

Therefore, we conclude that the limit sheaf  $\mathcal{E}_0$  is *asymptotically semistable*.

Note that there is nothing particular about  $\mathbb{P}^3$  here. Similar families of monads can be constructed over a wide class of projective Fano 3-folds  $W$ , say, using  $\mathcal{O}_W(1)$  and the embedding coordinates to form the maps.

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